SOLUTION OF SOME PROBLEMS OF HEAT AND MASS TRANSFER BY THE IMPEDANCE METHOD

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Inzhenerno-Fizicheskii Zhurnal, Vol. 12, No. 5, pp. 626–632, 1967

UDC 536.23+533.15

Heat and mass transfer in stationary binary gas mixtures in which periodic oscillations are induced is analytically investigated.

Steady-state problems of isothermal diffusion and heat conduction in stationary binary gas media are examined and the problem of heat and mass transfer with due regard to the concurrent effects of heat and mass transfer is solved. The solutions presented are obtained by the impedance method (the theory of passive quadripoles developed by Kovalenkov [3] and successfully applied by Grizodub [1]) in an examination of wave processes in pneumatic and hydraulic systems. The following method of calculating periodic wave processes of heat and mass transfer occurring simultaneously in a simple tube provides a means of analyzing harmonic oscillations of gas in thermal diffusion, thermal, and diffusion systems.

Isothermal mass transfer in nonreacting binary gas mixtures. The considered diffusion system is a bounded thermally insulated tube of constant cross section with a source of mass of substance of the 1st kind at one end ($x = 0$) and a consumer of this substance at the other end ($x = l$).

We solve the one-dimensional problem for the case where periodic oscillations of concentration and mass flux are imposed on the system (problem without initial conditions), i.e., when

$$c = C\exp[i(\omega_C\tau + \varphi_C)], \quad m = M\exp[i(\omega_C\tau + \psi_C)].$$

Mass transfer in a nonreacting binary gas mixture is described by the well-known equations [4]

$$-\frac{\partial c}{\partial x} = \frac{1}{\rho DF} m,$$
$$-\frac{\partial c}{\partial \tau} = \frac{1}{\rho F}\frac{\partial m}{\partial x}, \tag{1}$$

where ρ and D are assumed constant.

The last two equations for steady-state periodic oscillations can be put (if $\psi_C = \varphi_C$) in the following form:

$$-\frac{dC}{dx} = \frac{1}{\rho DF} M,$$
$$-i\omega_C C = \frac{1}{\rho F}\frac{dM}{dx}. \tag{2}$$

We assign boundary conditions in the form

$$\text{when } x = 0 \quad C = C_0,\ M = M_0;$$
$$\text{when } x = l \quad Z_C^l = \frac{C_l}{M_l}, \tag{3}$$

where Z_C^l is the end impedance (resistance), i.e., the total mass resistance offered by the medium to the propagation of mass from point $x = 0$ to point $x = l$ (in the general case Z_C^l is a complex quantity, since $\psi_C \neq \varphi_C$).

The solution of Eqs. (2) has the form

$$C = A\operatorname{ch}\gamma_C x + B\operatorname{sh}\gamma_C x,$$
$$M = -\frac{1}{Z_{xC}}(A\operatorname{sh}\gamma_C x + B\operatorname{ch}\gamma_C x), \tag{4}$$

where A and B are constants of integration; γ_C is the mass propagation constant:

$$\gamma_C = (1+i)\sqrt{\frac{\omega_C}{2D}}; \tag{5}$$

Z_{xC} is the mass characteristic resistance (impedance):

$$Z_{xC} = \frac{1-i}{\rho F\sqrt{2\omega_C D}}. \tag{6}$$

We find the constants of integration from the boundary conditions and on substituting them in (4), we obtain

$$C = C_0\operatorname{ch}\gamma_C x - M_0 Z_{xC}\operatorname{sh}\gamma_C x,$$
$$M = -Y_{xC}C_0\operatorname{sh}\gamma_C x + M_0\operatorname{ch}\gamma_C x, \tag{7}$$

where Y_{xC} is the mass characteristic admittance.

The initial mass impedance

$$Z_{0C} = C_0/M_0 \tag{8}$$

can be represented easily in the following form by using the transfer impedance [1]:

$$Z_{0C} = Z_{xC}\operatorname{th}(\gamma_C l + k_C),$$
$$k_C = \operatorname{arth} Z_C^l Y_{xC}. \tag{9}$$

Expression (9) connects the diffusion system with its boundary conditions.

Using relationships (7) and (9) we can write the obvious equation for the mass impedance at an arbitrary point x:

$$Z_C^x = Z_{xC}\operatorname{th}[\gamma_C(l-x) + k_C]. \tag{10}$$

The values of (8) and (9) for the initial mass impedance enable us to put expression (7) in the form

$$C = C_0[\operatorname{ch}\gamma_C x - \operatorname{cth}(\gamma_C l + k_C)\operatorname{sh}\gamma_C x],$$
$$C = M_0 Z_{xC}[\operatorname{th}(\gamma_C l + k_C)\operatorname{ch}\gamma_C x - \operatorname{sh}\gamma_C x]; \tag{11}$$

$$M = M_0\,[\mathrm{ch}\,\gamma_C x - \mathrm{th}\,(\gamma_C l + k_C)\,\mathrm{sh}\,\gamma_C x],$$

$$M = C_0 Y_{xC}\,[\mathrm{cth}(\gamma_C l + k_C)\,\mathrm{ch}\,\gamma_C x - \mathrm{sh}\,\gamma_C x]. \quad (12)$$

From formulas (11) and (12) we can find the amplitudes of the oscillations of concentration and mass flux at an arbitrary point x in steady-state conditions.

Steady-state heat conduction in a nonreacting binary gas mixture. We consider the thermal system. The difference between this system and the diffusion system considered above is the presence of a heat source (no mass source) at x = 0 and a heat sink at $x = l$. We will solve the one-dimensional problem of heat transfer for the case where periodic oscillations of temperature and heat flux are induced in the thermal system (problem without initial conditions), i. e., when

$$t = T\exp[i\,(\omega_T \tau + \varphi_T)], \quad q = Q\exp[i(\omega_T \tau + \psi_T)].$$

We assign the boundary conditions in the form

when x = 0 $T = T_0$ and $Q = Q_0$;

and at $x = l$ the end impedance, which is the total thermal resistance offered by the medium to the propagation of heat from point x = 0 to point $x = l$ (in the general case the thermal resistance is complex, since $\varphi_T \neq \psi_T$):

$$Z_T^l = T_l / Q_l\,.$$

The propagation of heat is described by the well-known equations [4]

$$-\frac{\partial t}{\partial x} = \frac{1}{\lambda F}\,q,$$

$$-\frac{\partial t}{\partial \tau} = \frac{1}{c_p \rho F}\,\frac{\partial q}{\partial x}, \quad (13)$$

where c_p, ρ, and λ—the specific heat, density, and thermal conductivity, respectively—are assumed to be constant.

Equations (13) for the propagation of heat are exactly analogous to Eqs. (1), which describe mass transfer. Hence, the solution of Eqs. (1) can be used for the solution of Eqs. (13), since even the boundary conditions for the two processes are analogous. Then, using expressions (5), (6), and (9)–(12) we can write

$$T = T_0\,[\mathrm{ch}\,\gamma_T x - \mathrm{cth}\,(\gamma_T l + k_t)\,\mathrm{sh}\,\gamma_T x],$$

$$T = Q_0 Z_{xT}\,[\mathrm{th}(\gamma_T l + k_t)\,\mathrm{ch}\,\gamma_T x - \mathrm{sh}\,\gamma_T x]; \quad (14)$$

$$Q = Q_0\,[\mathrm{ch}\,\gamma_T x - \mathrm{th}\,(\gamma_T l + k_t)\,\mathrm{sh}\,\gamma_T x],$$

$$Q = T_0 Y_{xT}\,[\mathrm{cth}\,(\gamma_T l + k_t)\,\mathrm{ch}\,\gamma_T x - \mathrm{sh}\,\gamma_T x]. \quad (15)$$

Here γ_T is the heat propagation constant:

$$\gamma_T = (1+i)\sqrt{\frac{\omega_T c_p \rho}{2\lambda}}\,; \quad (16)$$

$Z_{xT} = 1/Y_{xT}$ is the characteristic impedance,

$$Z_{xT} = \frac{1-i}{\sqrt{2\lambda F^2 c_p \rho\,\omega_T}}\,;$$

$$Z_T^x = Z_{xT}\,\mathrm{th}\,[\gamma_T (l - x) + k_t],$$

$$k_t = \mathrm{arth}\,Z_T^l\,Y_{xT}\,. \quad (17)$$

Thus, in the case of steady-state heat transfer the amplitudes of the temperature and heat flux oscillations at an arbitrary point x can be found from expressions (14) and (15).

Heat and mass transfer in stationary nonreacting binary gas mixtures. We consider the more general process of propagation of steady-state periodic oscillations of temperature, concentration, heat flux, and mass flux for a mixture of gases in the absence of external forces and friction forces.

The difference between this thermal diffusion system and the thermal and diffusion systems considered above is that at one end of the tube (x = 0) there is a mass and heat source, and at the other end there is a mass and heat sink.

The one-dimensional process of concurrent propagation of heat and mass is described by the differential equations [2]

$$m = -\rho\,DF\left(\frac{dc}{dx} + \frac{k_T}{t}\,\frac{dt}{dx}\right),$$

$$q = -\lambda F\,\frac{dt}{dx} - \frac{FDP}{\mu_0}\,\alpha\,\frac{dc}{dx}, \quad (18)$$

where $\mu_0 = \mu' c + \mu c'$, ρ, D, k_T/t, P, λ, and α are assumed to be constants.

We obtain the solution of the problem of propagation of periodic oscillations in a system in which periodic oscillations of concentration, temperature, mass flux, and heat flux are induced (problem without initial conditions).

The boundary conditions in the considered case will be as follows:

when $x = 0$, $T = T_0$, $Q = Q_0$, $C = C_0$, and $M = M_0$;

when $x = l$, $Z_{TC}^l = T_l/Q_l$ and $Z_{CT}^l = C_l/M_l$,

where the end impedance Z_{TC}^l (when $\varphi_T = \psi_T$) is the total thermal resistance offered by the medium to the propagation of heat, including heat transfer due to the concentration gradient, and the end impedance Z_{CT}^l (when $\varphi_C = \psi_C$) is the total mass resistance offered by the medium to the propagation of mass, including mass transfer due to the temperature gradient.

For steady-state periodic oscillations expressions (18) will take the form (when $\varphi_C = \psi_C$, $\varphi_T = \psi_T$, $\omega_C = \omega_T$)

$$M = -D_C\,\frac{dC}{dx} - D_T\,\frac{dT}{dx},$$

$$Q = -\lambda_T\,\frac{dT}{dx} - \lambda_C\,\frac{dC}{dx}, \quad (19)$$

where

$$D_C = \rho\,DF, \quad D_T = D_C\,\frac{k_T}{t}\,\exp(\varphi_C - \psi_T), \quad \lambda_T = \lambda F,$$

$$\lambda_C = \frac{PF\alpha}{\mu_0}\,\exp(\varphi_C - \psi_T).$$

We write total differentials for the concentration and temperature functions, assuming $c = c[m(x, \tau), q(x, \tau)]$, $t = t[m(x, \tau), q(x, \tau)]$:

$$dc = \left(\frac{\partial c}{\partial q}\right)_m dq + \left(\frac{\partial c}{\partial m}\right)_q dm,$$

$$dt = \left(\frac{\partial t}{\partial q}\right)_m dq + \left(\frac{\partial t}{\partial m}\right)_q dm.$$

If we assume that heat propagation does not affect the frequency of the concentration and mass flux and, conversely, diffusion does not affect the frequency of the temperature and heat flux, then for the case of steady-state periodic oscillations the last two equations can be put in the following form:

$$C = \left(\frac{C}{Q}\right)_{M=0} Q + \left(\frac{C}{M}\right)_{Q=0} M,$$

$$T = \left(\frac{T}{Q}\right)_{M=0} Q + \left(\frac{T}{M}\right)_{Q=0} M. \qquad (20)$$

The complex $(C/Q)_{M=0} = A_{11}(x)$ is the ratio of the amplitude of the oscillations of concentration of the first kind of substance, the change of which along the tube is due entirely to the heat flux (in the absence of mass flux), to the amplitude of the oscillations of this flux. This relationship is characterized by the Soret effect [2].

The value of the amplitude of the concentration oscillations in the absence of mass transfer ($M = 0$) is found from expression (19) with $M = 0$. Then function $A_{11}(x)$, in view of what has been said, takes the form

$$A_{11}(x) = \left[\frac{D_T(T_0 - T) + D_C C_0}{QD_C}\right]_{M=0}, \qquad (21)$$

where T and Q are the amplitudes of the oscillations of the gas temperature and heat flux, respectively, at an arbitrary point in the absence of mass transfer ($M = 0$).

To determine the amplitudes T and Q we can obviously use Eqs. (14) and (15). In view of what was said above, expression (21) can be written as

$$A_{11}(x) = \Big(D_T\{1 - [\operatorname{ch}\gamma_T x - \operatorname{cth}(\gamma_T l + k_t)\operatorname{sh}\gamma_T x]\} + D_C C_0/T_0\Big) \times \Big(D_C Y_{xT}[\operatorname{cth}(\gamma_T l + k_t)\operatorname{ch}\gamma_T x - \operatorname{sh}\gamma_T x]\Big)^{-1}, \qquad (22)$$

where

$$k_t = \operatorname{arth} Z^l_{TC} Y_{xT}.$$

The values of the heat propagation constant γ_T and the characteristic thermal impedance Z_{xT} are given by expressions (16) and (17). The complex $(C/M)_{Q=0} = A_{12}(x)$ is the mass impedance of the thermal diffusion system in the case of absence of heat transfer ($Q = 0$), i.e., when we are justified in using formula (10) for the mass impedance of the diffusion system

$$A_{12}(x) = Z_{xC}\operatorname{th}[\gamma_C(l - x) + k_C], \qquad (23)$$

where

$$k_C = \operatorname{arth} Z^l_{CT} Y_{xC}.$$

Constants γ_C and Z_{xC} are given exactly by expressions (5) and (6).

By similar arguments we can obtain values for the other two complexes:

$$\left(\frac{T}{Q}\right)_{M=0} = A_{21}(x) = Z_{xT}\operatorname{th}[\gamma_T(l - x) + k_t], \qquad (24)$$

$$\left(\frac{T}{M}\right)_{Q=0} = A_{22}(x) = \frac{\lambda_C\{1 - [\operatorname{ch}\gamma_C x - \operatorname{cth}(\gamma_C l + k_C)\operatorname{sh}\gamma_C x]\} + \lambda_T T_0/C_0}{\lambda_T Y_{xT}[\operatorname{cth}(\gamma_C l + k_C)\operatorname{ch}\gamma_C x - \operatorname{sh}\gamma_C x]}. \qquad (25)$$

Referring to (22)–(25), we obtain from (20)

$$C = A_{11}(x)Q + A_{12}(x)M,$$

$$T = A_{21}(x)Q + A_{22}(x)M.$$

The last two equations and the solution of Eqs. (19) form a system of four equations with four unknowns. The solution of this system is as follows:

$$M = \Big(T_0[A_{11}(x)a - xD_T] - C_0[A_{21}(x)a + xD_C]\Big) \times \Big(N(x)a - x[x + D_C A_{12}(x) + D_T A_{22}(x) + \lambda_T A_{21}(x) + \lambda_C A_{11}(x)]\Big)^{-1},$$

$$Q = \Big(C_0[A_{22}(x)a - x\lambda_C] - T_0[A_{12}(x)a + x\lambda_T]\Big) \times \Big(N(x)a - x[x + D_C A_{12}(x) + D_T A_{22}(x) + \lambda_T A_{21}(x) + \lambda_C A_{11}(x)]\Big)^{-1},$$

$$C = A_{11}(x)Q + A_{12}(x)M,$$

$$T = A_{21}(x)Q + A_{22}(x)M, \qquad (26)$$

where

$$a = D_C\lambda_T - \lambda_C D_T,$$

$$N(x) = A_{11}(x)A_{22}(x) - A_{12}(x)A_{22}(x).$$

Thus, the amplitudes of the oscillations of the heat and mass fluxes and the amplitudes of the concentration and temperature oscillations at an arbitrary point x of the thermal diffusion system are given by expressions (26), in which the functions are found from formulas (22)–(25).

Example. Find the distribution of the amplitude of the oscillations of carbon dioxide concentration in a binary mixture of CO_2 and NO_2 of the diffusion system considered at the beginning of this paper, if the known values are: $C_0 = 0.8$, $\rho = 1$ kg/m^3, $F = 10^{-4}$ m^2, $l = 0.6$ m, $Z^l_C = 7 \cdot 10^6$ sec/kg, $D = 9.6 \cdot 10^{-6}$ m^2/sec, $\omega_C = 0.192 \cdot 10^{-2}$ Hz.

Solution. Substituting the known values of the problem in expressions (5) and (6) and using the tables for hyperbolic functions we calculate the propagation constant

$$\gamma_C = (1 + i)\sqrt{\frac{0.192 \cdot 10^{-2}}{2 \cdot 9.6 \cdot 10^{-6}}} = 10(1 + i) \quad 1/\text{m},$$

the characteristic impedance

$$Z_{xC} = \frac{1 - i}{\sqrt{2 \cdot 9.6 \cdot 10^{-6} \cdot 1 \cdot 10^{-8} \cdot 0.192 \cdot 10^{-2}}} = 5.2 \cdot 10^7 (1 - i) \text{ sec/kg},$$

the quantity

$$\operatorname{cth}(\gamma_C l + k_C) = \operatorname{cth}[10(1+i)\cdot 0.6 + \operatorname{arth} 7\cdot 10^6 \frac{1}{5.2\cdot 10^7(1-i)} \cong 1.$$

Then the first expression of relationship (11) can be put in the following form:

$$C = 0.8[\operatorname{ch} 10(1+i)x - 1\cdot \operatorname{sh} 10(1+i)x] = 0.8 \exp(-10x)(\cos 10x + \sin 10x).$$

For instance, for x = 0.1 m, C = 0.272.

NOTATION

c and C are the concentration and amplitude of concentration oscillations, respectively, of 1st kind of gas; m and M are the mass flux and amplitude of mass flux oscillations for 1st kind of gas; t and T are the temperature of mixture and amplitude of temperature oscillations; q and Q are the heat flux and amplitude of heat flux oscillations; P is the pressure; ρ is the density; λ is the thermal conductivity; c_p is the specific heat; D is the diffusion coefficient; k_T is the thermal diffusion coefficient; α is the thermal diffusion factor; μ and μ' are the molecular weights of 1st and 2nd kinds of gas; c' is the concentration of 2nd kinds of gas; ω_T is the frequency of oscillations of temperature and heat flux; ω_C is the frequency of oscillations of concentration and mass flux; ψ_T and φ_T are the initial phases for heat flux and temperature; ψ_C and φ_C are the initial phases for mass flux and concentration; F is the cross-sectional area; x is the coordinate; τ is time; $i = (-1)^{1/2}$.

REFERENCES

1. Yu. N. Grizodub, ZhTF, **21**, no. 5, 1951.
2. S. de Groot and P. Mazur, Nonequilibrium Thermodynamics [Russian translation], Izd. Mir, Moscow, 1964.
3. V. I. Kovalenkov, Izv. AN SSSR, OTN, no. 12, 1945.
4. A. V. Luikov and Yu. A. Mikhailov, Theory of Energy and Mass Transfer [in Russian], Izd. AN BSSR, 1959.

18 October 1966

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